Snow College Mathematics Contest

April 3, 2007 **Senior division: grades 10–12** **Form: A**

Please read all instructions on this page very carefully.

1. Leave this booklet closed until you are instructed to begin.
2. Go ahead now and fill in the box at the top of your answer sheet. Print your name clearly, put your phone number in the "ID#" blank, spell out your school in the "class" blank, and put your year in school in the "sec" blank. Put your test version (Form A) in the "test no." blank. Also use a #2 (or HB or soft) pencil to bubble in your name on the left side of the answer sheet.
3. This is a two hour examination consisting of 40 multiple choice problems. Avoid random guessing as there is a penalty for wrong answers. There is no penalty for leaving a question blank. The formula for scoring the test is Score $= 4R - W$ where R and W denote the number right and wrong respectively. The possible scores range from -40 to 160.
4. In the event of a tie, the person with the largest number of the following five problems correct will be declared the winner: 11, 21, 23, 28, 38. Any further ties will be broken by a coin toss.
5. When the test begins, bubble in the single best answer to each question you choose to answer clearly on the answer sheet. Use #2 (or soft) pencil. Completely erase any incorrect answers.
6. The sketches that accompany the problems are not necessarily drawn to scale.
7. No calculators are allowed.
8. Do not talk or disrupt other test takers during the exam. Cell phones must be OFF.
9. Please raise your hand if you need scratch paper; a proctor will assist you.
10. The proctors have been advised to answer no questions pertaining to the exam.
11. While we recommend you stay and recheck your answers if you have time, you may leave if you finish early (if you do, turn your answer sheet in and leave quietly). After the two hour time limit is up the proctors will call for your answer sheets. Hand them in promptly.

After the test:

1. You may keep this test booklet.
2. If you RSVP'd to spend time with one of our science departments for lunch, please meet them in the science building; otherwise lunch may be purchased at the Snow College Cafeteria or downtown. In any event, you should plan to be back at the LDS Institute by 1:30 p.m. for the scores and presentation of the awards.
3. The top three scorers from each classification of school will receive full tuition scholarships to Snow College. Other prizes will be awarded to other individuals.
4. Thanks for coming. Your instructors will be happy to work the problems for you, and they will also be given copies of your answer sheets.

Snow College Mathematics Contest

April 3, 2007 **Senior division: grades 10-12** **Form: A** **key**

Bubble in the single best answer to each question.

1. If you use the eight digits 1, 2, 3, 4, 5, 6, 7, and 9 each once and only once to form four 2-digit prime numbers, what will be the sum of the four prime numbers you created?

(A) 170

(B) 175

(C) 180

(D) 185

(E) 190

There are multiple possiblities but we do know that no 2-digit even numbers are prime; nor are numbers ending in 5. Therefore the digits 2, 4, 6, and 5 must be in the tens' place. That leaves 1, 3, 7, and 9 in the ones' place.

2. One mnemonic to remember the names of the notes on the lines of the treble clef is "Every Good Boy Does Fine." What is the negation of "Every good boy does fine"?

(A) "Some good boys do fine."

(B) 'Those who do not do fine are not good boys."

(C) "Some who do fine are not good boys."

(D) "There is a good boy who does not do fine."

(E) "Everyone who does fine is a good boy."

The negation of "For all something, p" is the statement "There exists something such that $\neg p$".

3. If $f(x) = \dfrac{x-1}{x-2}$, what is $f^{-1}(4)$? (Here f^{-1} is the inverse function of f.)

(A) 3

(B) -3

(C) 3/2

(D) 2/3

(E) 7/3

$f^{-1} = \frac{2x-1}{x-1}$.

4. An equilateral triangle and a regular hexagon have the same perimeter. What is the ratio of the area of the triangle to the area of the hexagon?

(A) 1/2

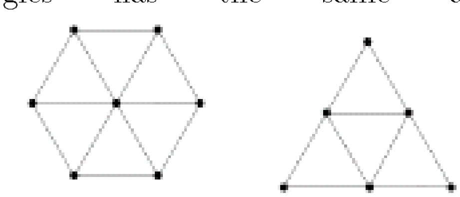
(B) 2/3

(C) 3/4

(D) $\sqrt{2}/2$

(E) $\sqrt{3}/3$

Each of the small triangles has the same area.



5. What is the integer n for which $5^n + 5^n + 5^n + 5^n + 5^n = 5^{25}$?

(A) 4

(B) 5

(C) 6

(D) 10

[(E)] 24

If $5^n + 5^n + 5^n + 5^n + 5^n = 5 \times 5^n = 5^{n+1} = 5^{25}$, then $n + 1 = 25$ and $n = 24$.

6. If Jo wants to mail a package which requires \$1.53 in postage, and has only 5-cent and 8-cent stamps, what is the smallest number of stamps she could use to total exactly \$1.53?

(A) 24

(B) 23

[(C)] 21

(D) 14

(E) none of these

You want as many 8-cent stamps as possible but still leaving a difference to 153 that is a multiple of 5. Try 19 8-cent stamps; that doesn't leave a difference that is a multiple of 5. Try 18 8-cent stamps; likewise. But $8 \times 16 = 128$ which leaves a difference of 25. So 16 8-cent stamps plus 5 5-cent stamps.

7. A *derangement* of n distinct symbols which have some natural order is a permutaion in which no symbol is in its correct position. The number of derangments for n symbols is denoted D_n. For example there is just one derangement of the symbols $1, 2$ (namely $2, 1$), so $D_2 = 1$. What is D_4?

[(A)] 9

(B) 10

(C) 11

(D) 12

(E) none of these

$$D_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right)$$

2341	2413	2143
3142	3412	3421
4123	4312	4321

8. Which of these numbers is the greatest?

[(A)] $2^{(3^4)}$

(B) $4^{(3^2)}$

(C) $(8^4)^2$

(D) $(16^8)^2$

(E) $8^{(4^2)}$

The numbers are 2^{81}, 2^{18}, 2^{24}, 2^{64}, and 2^{48}.

9. Sue had walked halfway from home to school when she realized she was late. She ran the rest of the way to school. She ran 3 times as fast as she walked. Sue took 6 minutes to walk halfway to school. How many minutes did it take Sue to get from home to school?

(A) 7

(B) 7.3

(C) 7.7

(D) 8

(E) 8.3

Covering the same distance three times as fast takes one-third the time. So Sue ran for 2 minutes. Her total time was $6 + 2 = 8$ minutes.

10. The composite of two functions f and g is denoted by $f \circ g$ and defined by $(f \circ g)(x) = f(g(x))$. When $f(x) = \dfrac{6x}{x-1}$ and $g(x) = \dfrac{5x}{x-2}$ which one of the following is equal to $(f \circ g)(x)$?

(A) $\dfrac{4-x}{x-2}$

(B) $\dfrac{30x}{5x+2}$

(C) $\dfrac{x-2}{4x+2}$

(D) $\dfrac{15x}{2x+1}$

(E) $\dfrac{x}{5x-2}$

$$f(g(x)) = \frac{6\left(\frac{5x}{x-2}\right)}{\frac{5x}{x-2} - 1} = \frac{6\left(\frac{5x}{x-2}\right)}{\frac{5x-x+2}{x-2}} = \frac{30x}{4x+2}$$

11. Three positive numbers a, b, and c form an arithmetic progression. Oddly enough, if you increase a by 1 you get a geometric progression. Even more oddly, if you increase c by 2 you also get a geometric progression (with the original a and b). What is the value of b?

(A) 8

(B) 9

(C) 10

(D) 12

(E) 18

We need to solve the system of equations

$$\begin{aligned} b - a &= c - b && (1) \\ \frac{b}{a+1} &= \frac{c}{b} && (2) \\ \frac{b}{a} &= \frac{c+2}{b} && (3) \end{aligned}$$

Cross multiply in Equations 2 and 3 and then eliminate b^2 to get $c = 2a$. Solve for c in Equation 1 and set the two expressions for c equal to each other to find $b = \frac{3}{2}a$. Plug $b = \frac{3}{2}a$ and $c = 2a$ into Equation 2 to find that $a = 8$.

12. If $x + 2y = 84 = 2x + y$, what is the value of $x + y$?

(A) 56

(B) 62

(C) 66

(D) 74

(E) 84

Since $x + 2y = 84$ and $2x + y = 84$, then add the two equations together to obtain $3x + 3y = 168$. Divide by 3 to get $x + y = 56$.

Or, since the two equations are identical when x is replaced with y and vice versa, then $x = y$. So $3x = 84 \Rightarrow x = 28$.

13. If $\overline{AB}$ is the diameter of a circle and C is on the circle such that $\overline{AC} = 10$ and $\overline{BC} = 16$ then what is the area of the circle?

(A) 89π

(B) $2\pi\sqrt{89}$

(C) 178π

(D) 356π

(E) none of these

The diameter and two chords form a right triangle so the diameter is $\sqrt{10^2 + 16^2} = \sqrt{356}$. The radius is half that. $A = \pi r^2 = \pi(\sqrt{356}/2)^2$.

14. Suppose all of the vehicles traveling on a certain interstate highway have either 18 wheels on 5 axles or 4 wheels on 2 axles. In a five minute period, 224 wheels on 88 axles pass by. How many vehicles passed by during this period?

(A) 18

(B) 23

(C) 29

(D) 31

(E) 35

If x is the number of 18-wheeled vehicles and y is the number of 4-wheeled vehicles then we must solve the simultaneous equations

$$\begin{aligned} 18x + 4y &= 224 \\ 5x + 2y &= 88 \end{aligned}$$

The solution is $x = 6$ and $y = 29$.

15. In some military applications, angles are measured in mils. The arc length cut off on a circle of radius 1000 meters, by a 1-mil angle at its center, is 1 meter. How many mils are in a 45° angle?

(A) $\frac{1000}{45}$

(B) 250

(C) 250π

(D) 500π

(E) 45 000

Since $\theta = s/r$ then
$1\,\text{mil} = 1\,\text{m}/1000\,\text{m} = 0.001\,\text{rad}$.

$$45^\circ = 45^\circ \left(\frac{\pi\,\text{rad}}{180^\circ}\right)\left(\frac{1\,\text{mil}}{0.001\,\text{rad}}\right) = 250\pi\,\text{mil}$$

16. Convert the base three number 2102 to base ten.

  (A) 65
  (B) 92
  (C) 73
  (D) 1121
  (E) 81

  $2102 = 2\times3^3+1\times3^2+0\times3^1+2\times3^0 = 2\times27+1\times9+2\times1$

17. What is the reciprocal of $2+\mathrm{i}$?

  (A) $2+\mathrm{i}$
  (B) $2-\mathrm{i}$
  (C) $\frac{2}{5}+\frac{\mathrm{i}}{5}$
  (D) $\frac{2}{5}-\frac{\mathrm{i}}{5}$
  (E) none of the above

$$\frac{1}{(2+\mathrm{i})}\frac{(2-\mathrm{i})}{(2-\mathrm{i})}=\frac{2-\mathrm{i}}{4+1}$$

18. Topology is the study of the properties that are preserved through deformations, twistings, and stretchings of objects. Tearing is not allowed. A circle is topologically equivalent to an ellipse.
A cocoa mug with one handle is topologically equivalent to which of the following?

  (A) whole doughnut
  (B) doughnut hole
  (C) disk-shaped cookie
  (D) wedge-shaped piece of cake
  (E) fortune cookie

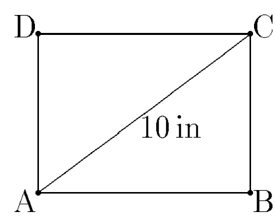
  Only the doughnut has exactly one hole like the mug.

19. What is $\cot(\sin^{-1}(\frac{3}{5}))$?

  (A) $\frac{5}{3}$
  (B) $\frac{5}{4}$
  (C) $\frac{4}{5}$
  (D) $\frac{4}{3}$
  (E) $\frac{3}{4}$

  SOH CAH TOA says that we have a right triangle with sides 3, 4, and 5. If $\sin\theta=\frac{3}{5}$, then the adjacent side must be 4. $\tan\theta=\frac{3}{4}$.

20. The area of the rectangle $ABCD$ is $48\,\text{in}^2$ and the length of its diagonal is 10 in. What is the perimeter of the rectangle?



  (A) 8 in
  (B) 12 in
  (C) 18 in
  (D) 20 in
  (E) 28 in

  Let x be the length of side AB and y be the length of side BC. Then $xy=48$ and $x^2+y^2=100$. The factors of 48 suggest $x=8$ and $y=6$ and those choices also give the correct proportions for a 3-4-5 right triangle.

21. Matrix multiplication is not commutative in general. Given matrices A and B, what is $\det(AB - BA)$?

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$$

(A) -19
(B) -29
(C) 19
(D) 29
(E) None of these.

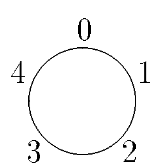
Multiplying the matrices gives

$$AB = \begin{bmatrix} 5 & 13 \\ 5 & 11 \end{bmatrix} \qquad BA = \begin{bmatrix} 10 & 10 \\ 7 & 6 \end{bmatrix}$$

$$AB - BA = \begin{bmatrix} -5 & 3 \\ -2 & 5 \end{bmatrix}$$

22. Consider a finite arithmetic on five elements: 0, 1, 2, 3, and 4, such that addition wraps around; for example, $4 + 3 = 2$. What is the multiplicative inverse (*i.e.*, reciprocal) of 3? (Hint: a multiplication table is helpful.)

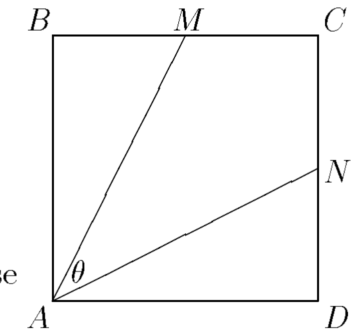
(A) 0
(B) 1
(C) 2
(D) 3
(E) 4



$3 + 3 = 2 \times 3 = 1$, so $3^{-1} = 2$.

23. $ABCD$ is a square and M and N are the midpoints of BC and CD respectively. What is $\sin\theta$?

(A) $\frac{\sqrt{5}}{5}$
(B) $\frac{3}{5}$
(C) $\frac{\sqrt{10}}{5}$
(D) $\frac{4}{5}$
(E) None of these



Say the side of the square has length 1, then $AN = AM = \sqrt{5}/2$ and $DN = 1/2$. Call $\angle MAD$ β, and call $\angle NAD$ α.

$$\begin{aligned} \sin\theta &= \sin(\beta - \alpha) \\ &= \sin\beta\cos\alpha - \cos\beta\sin\alpha \\ &= \frac{1}{\sqrt{5}/2}\frac{1}{\sqrt{5}/2} - \frac{1/2}{\sqrt{5}/2}\frac{1/2}{\sqrt{5}/2} \\ &= \frac{4}{5} - \frac{1}{5} \\ &= \frac{3}{5} \end{aligned}$$

24. How many ordered triples (a, b, c) of non-zero real numbers have the property that each number is the product of the other two?

(A) 1
(B) 2
(C) 3
(D) 4
(E) 5

$(1, 1, 1)$, $(1, -1, -1)$, $(-1, 1, -1)$, $(-1, -1, 1)$ This can also be seen by simultaneously graphing in 3-D $z = xy$, $y = xz$, and $x = yz$.

25. Suppose $y > 0$, $x > y$, and $z \neq 0$. Which inequality is always correct?

(A) $x + z > y - z$

(B) $xy > yz$

(C) $\frac{x}{z} > \frac{y}{z}$

(D) $xz^2 > yz^2$

(E) none of the above

Options A and C are incorrect when $z < 0$; there are myriad examples of B being incorrect. Option D is correct even when $z < 0$.

26. How many of the following four numbers are integers?

$$\sin^2\left(\frac{\pi}{6}\right)+\cos^2\left(\frac{\pi}{3}\right),\ e^{\tan(0)},\ \cos^{-1}(0),\ \ln(e^3)$$

(A) 0

(B) 1

(C) 2

(D) 3

(E) 4

The first one would be if the argument of $\sin^2$ and $\cos^2$ were the same. $e^{\tan(0)} = 1$, $\cos^{-1}(0) = \pi/2\,\text{rad}$, $\ln(e^3) = 3$

27. A special deck of 20 cards contains 10 red, 7 blue, and 3 green cards. If 2 cards are selected at random (without replacement), what is the probability that both cards are the same color?

(A) $\frac{1}{3}$

(B) $\frac{69}{200}$

(C) $\frac{79}{200}$

(D) $\frac{69}{190}$

(E) $\frac{79}{190}$

$$\left(\frac{10}{20}\cdot\frac{9}{19}\right)+\left(\frac{7}{20}\cdot\frac{6}{19}\right)+\left(\frac{3}{20}\cdot\frac{2}{19}\right)$$

28. What is the number of positive integers less than 1000 divisible by neither 5 nor 7?

(A) 630

(B) 658

(C) 686

(D) 33

(E) none of these

$$1000 - \frac{1000}{5} - \frac{1000}{7} + \frac{1000}{5\cdot 7}$$

29. If a dart is thrown at a circular dart board and hits it at a random point, what is the probability that it lands closer to the center than the outside edge?

(A) $\frac{1}{2}$

(B) $\frac{1}{4}$

(C) $\frac{1}{8}$

(D) $\frac{1}{25}$

(E) none of the above

A circle of radius $R/2$ has $1/4$ the area of a circle of radius R because area scales with R^2.

30. Find the number of units that produce a maximum revenue, $R = 95x - 0.1x^2$, where R is the total revenue in dollars and x is the number of units sold.

(A) 716 units

(B) 642 units

(C) 550 units

(D) 475 units

(E) none of the above

To maximize, take the derivative and set equal to zero. $95 - 0.2x = 0$. Then multiply by 5 to solve for x.

31. How many solutions does the trigonometric equation $\dfrac{\sin x}{1+\cos x} = 1$ have in the interval $[0, 2\pi]$? (Hint: it is not the same answer as the number of solutions to $\dfrac{\cos x}{1+\sin x} = 1$ in the same interval.)

(A) 0

(B) 1

(C) 2

(D) 4

(E) infinitely many

The solution is $x = \pi/2$. This is most easily seen by a quick sketch of $\frac{\sin x}{1+\cos x}$. $x = \pi$ is not a solution because the denominator is undefined there.

For the equation $\dfrac{\cos x}{1+\sin x} = 1$ there are solutions at 0 and 2π.

32. There are six cards colored red, green, or blue on one side. The other side of each card has one of the symbols $\circledcirc$, $\bowtie$, or $\divideontimes$ on it. Consider the statement: "Every green card has a $\divideontimes$ on the other side." To prove or disprove the statement, which of the following cards must be turned over and checked?

red	$\circledcirc$	blue
1	2	3
$\bowtie$	green	$\divideontimes$
4	5	6

(A) card 5 only

(B) cards 5 and 6 only

(C) cards 2, 4, and 5 only

(D) cards 2, 5, and 6 only

(E) cards 2, 4, 5, and 6 only

Card 5 must be checked to see if it has $\divideontimes$ on the other side. Cards 2 and 4 must also be checked to see if they are green on the other side.

33. Given the repeating decimals $x = 0.\overline{23}$ and $y = 1.\overline{4}$, then what is $x + y$?

(A) $\frac{167}{98}$

(B) $\frac{166}{99}$

(C) $\frac{168}{101}$

(D) $\frac{168}{98}$

(E) $\frac{167}{101}$

$z = x + y = 1.\overline{67}$

$$\begin{array}{rcl} 100z &=& 167.\overline{67} \\ z &=& 1.\overline{67} \\ \hline 99z &=& 166 \end{array}$$

34. How much is the total surface area of a cube, with edge e, increased if the length of each edge is increased by 2 units?

(A) $6(3e + 1)$

(B) $12(e + 1)$

(C) $12(e + 6)$

(D) $24(e + 1)$

(E) none of the above

For the original cube $A = 6e^2$. If each edge is increased by 2 then the new surface area is $A' = 6(e + 2)^2$. The increase is $A' - A$.

35. If the greatest common divisor of integers m and n is q then what is the least common multiple of m and n?

(A) mnq

(B) $\frac{mn}{q}$

(C) $\frac{q}{mn}$

(D) mn

(E) none of the above

$\text{lcm}(m, n) \cdot \gcd(m, n) = mn$

36. Which of the following statements are true?

(i) The sum of two rational numbers must be rational.

(ii) The sum of two irrational numbers must be irrational.

(iii) The product of two rational numbers must be rational.

(iv) The product of two irrational numbers must be irrational.

(A) only(i) and (iii)
(B) only (i) and (ii)
(C) only (i), (ii), and (iii)
(D) all of them
(E) none of them

(i) and (iii) are true because the rational numbers are closed under addition and multiplication. The irrational numbers are not; for example, $\sqrt{5} + (6 - \sqrt{5}) = 6$ and $\sqrt{7} \times \sqrt{7} = 7$.

37. Consider the graphs of cubic polynomials. Which of the statements must be true?

(i) They must have an x-intercept.

(ii) They must have a y-intercept.

(iii) They must have a local minimum.

(A) only(i) and (ii)
(B) only (i) and (iii)
(C) only (ii) and (iii)
(D) all of them
(E) none of them

(iii) is not necessarily true; for example, $y = x^3$.

38. What integer n satisfies the following?

$$\log_{10}(16!) - \log_{10}(14!) = 1 + \log_{10}(n!)$$

(A) 3
(B) 4
(C) 5
(D) 6
(E) none of the above

$$\log_{10}(16!) - \log_{10}(14!) - \log_{10}(n!) = 1$$

$$\log_{10}\left(\frac{16!}{14!\,n!}\right) = 1$$

$$\frac{16!}{14!\,n!} = 10$$

$$\frac{16 \cdot 15}{10} = n!$$

$$24 = n!$$

39. A square and a circle have the same area. If the length of the side of the square is tripled and the radius of the circle is tripled, what is the ratio of the area of the new circle to the area of the new square?

(A) $\frac{3}{2}$
(B) π
(C) $\frac{1}{3}\pi$
(D) $\frac{1}{3}$
(E) 1

Both areas scale with the square of the length (side or radius), so since they were equal areas to begin with and the length was scaled by a factor of 3 in each case then the new areas will also be equal.

40. How many of the following could be the intersection of a plane and the surface of a cube: empty set, line segment, triangle, quadrilateral, pentagon, hexagon?

(A) 2

(B) 3

(C) 4

(D) 5

(E) 6

All of them are possible in various orientations of the cube with respect to the plane.